Numerik 2 – Übung08 – Georg Kuschk

1.)

$$\frac{1}{5}\sum_{i=0}^{4}\left(P(t_i)-f_i\right)^2=\frac{1}{5}\left\|A\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}-f\right\|_2^2$$

$$P(t_i)=a+bt_i+ct_i^{\,2}$$

$\Rightarrow$

$$A=\begin{pmatrix}1&0&0\\1&1&1\\1&2&4\\1&3&9\\1&4&16\end{pmatrix}\quad,\quad f=\begin{pmatrix}0\\1\\2\\1\\0\end{pmatrix}$$

$\Rightarrow$ Lösung des linearen Ausgleichsproblems :

1.) Über Normalgleichung :

$$A^T A\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=A^T\cdot f$$

$$\begin{pmatrix}5&10&30\\10&30&100\\30&100&354\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}4\\8\\18\end{pmatrix}$$

2.) Über orthogonale Transformationen (bessere Kondition)

$$\left\|A\cdot\begin{pmatrix}a\\b-f\\c\end{pmatrix}\right\|_2=\left\|Q^T A\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}-Q^T f\right\|=\left\|\begin{pmatrix}R\\---\\0\end{pmatrix}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}-\begin{pmatrix}\tilde{f}_1\\--\\\tilde{f}_2\end{pmatrix}\right\|$$

Lösung von $R\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=\tilde{f}_1$

$\Rightarrow\qquad P(t)=-0.057+1.714t-0.429t^2$

(Matlab-Befehl zum Lösen von LGS : (a b c)=A\b)

2.a)

per Induktion :

<u>$n=1$:</u>

$$\sin\left(\frac{t-t_1}{2}\right)\sin\left(\frac{t-t_2}{2}\right)$$
$$=-\frac{1}{2}\cos\left(t-\frac{t_1+t_2}{2}\right)+\frac{1}{2}\cos\left(\frac{t_1-t_2}{2}\right)$$
$$=\underbrace{\frac{1}{2}\cos\left(\frac{t_1-t_2}{2}\right)}_{=a_0}-\frac{1}{2}\left(\cos t\cdot\underbrace{\cos\left(\frac{t_1+t_2}{2}\right)}_{=-2a_1}\right)-\frac{1}{2}\left(\sin t\cdot\underbrace{\sin\left(\frac{t_1+t_2}{2}\right)}_{=-2b_1}\right)$$
$$=a_0+a_1\cos t+b_1\sin t$$

<u>Induktionsschritt :</u>

$$\prod_{k=1}^{2(n+1)}\sin\left(\frac{t-t_k}{2}\right)=\frac{1}{2}a_0+\sum_{j=1}^{n}(\ldots)\cdot(\tilde{a}_0+\tilde{a}_1\cos t+\tilde{b}_1\sin t)$$
$$(\tilde{a}_0+\tilde{a}_1\cos t+\tilde{b}_1\sin t)=\sin\left(\frac{t-t_{2n+1}}{2}\right)\cdot\sin\left(\frac{t-t_{2n+2}}{2}\right)$$

$$\cos(jt)\cdot(\tilde{a}_0+\tilde{a}_1\cos t+\tilde{b}_1\sin t)$$
$$=(\tilde{a}_0\cos(jt)+\tilde{a}_1\cdot\frac{1}{2}\left(\cos((j+1)t)+\cos((j-1)t)\right)$$
$$+\tilde{b}_1\cdot\frac{1}{2}\left(\sin((j+1)t)-\sin((j-1)t)\right))$$

$$\sin(jt)\cdot(\tilde{a}_0+\tilde{a}_1\cos t+\tilde{b}_1\sin t)$$
$$=(\tilde{a}_0\sin(jt)+\tilde{a}_1\cdot\frac{1}{2}\left(\sin((j+1)t)+\sin((j-1)t)\right)$$
$$+\tilde{b}_1\cdot\frac{1}{2}\left(-\cos((j+1)t)+\cos((j-1)t)\right))$$

$\Rightarrow$

$$\prod_{k=1}^{2n+2}\sin\left(\frac{t-t_k}{2}\right)=\frac{1}{2}\hat{a}_0+\sum_{j=1}^{n}(\hat{a}_j\cos t+\hat{b}_j\sin t)$$

2.b)

$$T_j(t_l)=d_{j,l}$$
$\Rightarrow$
$$\Phi_{2n+1}(t_l)=f_l$$

Mit 2.a) folgt hieraus die Behauptung.

3.a)

Erinnerung : $\cos(jt_k) + i\sin(jt_k) = e^{ijt_k}$

$$
\begin{aligned}
f(t) &= \sum_{k=0}^{2m}\left(\cos(jt_k) + i\sin(jt_k)\right) \\
&= \sum_{k=0}^{2m} e^{ijt_k} \\
&= \sum_{k=0}^{2m} e^{\frac{2p\cdot ijk}{2m+1}} \\
&= \sum_{k=0}^{2m} e^{\frac{2p\cdot i\cdot(j \bmod 2m+1)\cdot k}{2m+1}}
\end{aligned}
$$

$\Rightarrow$

$$\sum_{k=0}^{2m}\cos(jt_k) = \mathrm{Re}\left(f(t)\right) = (2m+1)\cdot h(j)$$

$$\sum_{k=0}^{2m}\sin(jt_k) = \mathrm{Im}\left(f(t)\right) = 0$$

3.b)

- $$\sum_{l=0}^{2m}\left(\sin(jt_l)\cdot\sin(kt_l)\right) = \frac{1}{2}\sum_{l=0}^{2m}\cos((j-k)t_l) - \frac{1}{2}\sum_{l=0}^{2m}\cos((j+k)t_l)$$
  $$= \frac{2m+1}{2}\left(h(j-k) - h(j+k)\right)$$

- $$\sum_{l=0}^{2m}\left(\cos(jt_l)\cdot\cos(kt_l)\right) = \frac{1}{2}\sum_{l=0}^{2m}\cos((j+k)t_l) + \frac{1}{2}\sum_{l=0}^{2m}\cos((j-k)t_l)$$
  $$= \frac{2m+1}{2}\left(h(j+k) + h(j-k)\right)$$

- $$\sum_{l=0}^{2m}\left(\cos(jt_l)\cdot\sin(kt_l)\right) = \frac{1}{2}\sum_{l=0}^{2m}\sin((j+k)t_l) - \frac{1}{2}\sum_{l=0}^{2m}\sin((j-k)t_l) \quad = 0$$