Analysis 3 – Serie04 – Georg Kuschk

4.1)

gegeben : $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$, $n = 0,1,\ldots$

$\Phi_n(x) = \sqrt{n+\frac{1}{2}}P_n(x)$ im Intervall [-1,+1]

zu zeigen : $\int_{-1}^{1} \Phi_n(x)\cdot\Phi_m(x)dx = d_{n,m}$

Setze zunächst $f_n(x) := \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$

D.h. $\int_{-1}^{1} \Phi_n(x)\cdot\Phi_m(x)dx = \left(n+\frac{1}{2}\right)\cdot\frac{1}{2^n n!}\cdot\frac{1}{2^m m!}\cdot\int_{-1}^{1} f_n(x) f_m(x)dx$ (*1)

o.B.d.A. sei $m \geq n$, d.h. m = n+k mit $k \geq 0$ und $k \in N$

$$\int_{-1}^{1} f_n(x) f_m(x)dx = \int_{-1}^{1} \underbrace{\frac{d^n}{dx^n}\left[(x^2-1)^n\right]}_{u}\cdot\underbrace{\frac{d^{n+k}}{dx^{n+k}}\left[(x^2-1)^{n+k}\right]}_{v'}dx$$ partielle Integration anwenden

$$= \frac{d^n}{dx^n}\left[(x^2-1)^n\right]\cdot\underbrace{\left[\frac{d^{n+k-1}}{dx^{n+k-1}}\left[(x^2-1)^{n+k}\right]\right]_{-1}^{1}}_{=0} - \int_{-1}^{1}\frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)^n\right]\cdot\frac{d^{n+k-1}}{dx^{n+k-1}}\left[(x^2-1)^{n+k}\right]dx$$

$$= -\int_{-1}^{1}\frac{d^{n+1}}{dx^{n+1}}\left[(x^2-1)^n\right]\cdot\frac{d^{n+k-1}}{dx^{n+k-1}}\left[(x^2-1)^{n+k}\right]dx = \ldots = (-1)^n\int_{-1}^{1}\frac{d^{2n}}{dx^{2n}}\left[(x^2-1)^n\right]\cdot\frac{d^k}{dx^k}\left[(x^2-1)^{n+k}\right]dx$$

wobei $\frac{d^{2n}}{dx^{2n}}\left[(x^2-1)^n\right] = \frac{d^{2n}}{dx^{2n}}x^{2n} = (2n)!$

$$\Rightarrow \int_{-1}^{1} f_n(x) f_{n+k}(x)dx = (-1)^n(2n)!\int_{-1}^{1}\frac{d^k}{dx^k}\left[(x^2-1)^{n+k}\right]dx$$

Fallunterscheidung von k :

<u>1.Fall :</u> $k > 0$ (d.h. $n \neq m$)

$$\Rightarrow \int_{-1}^{1} f_n(x) f_{n+k}(x)dx = (-1)^n(2n)!\left[\frac{d^{k-1}}{dx^{k-1}}(x^2-1)^{n+k}\right]_{-1}^{1} = 0$$

D.h. $\int_{-1}^{1} \Phi_n(x)\cdot\Phi_m(x)dx = 0$ für $n \neq m$

<u>2.Fall :</u> $k = 0$ (d.h. $n = m$)

$$\Rightarrow \int_{-1}^{1} f_n(x) f_{n+k}(x)dx = (-1)^n(2n)!\int_{-1}^{1}(x^2-1)^n dx$$

Berechnung von $\int_{-1}^{1}(x^2-1)^n dx$ durch Rekurrenzauflösung

Setze $I_n := \int_{-1}^{1}(x^2-1)^n dx$

partielle Integration anwenden mit $u = (x^2-1)$ und $v' = 1$

$$\Rightarrow \int_{-1}^{1}(x^2-1)^n dx = (x^2-1)^n x\Big|_{-1}^{1} - \int_{-1}^{1} 2xn(x^2-1)^{n-1} x dx$$

$$= -2n\int_{-1}^{1} x^2(x^2-1)^{n-1} dx = -2n\int_{-1}^{1}(x^2-1+1)(x^2-1)^{n-1} dx$$

$$= -2n\left(\int_{-1}^{1}(x^2-1)^n dx + \int_{-1}^{1}(x^2-1)^{n-1} dx\right) = -2n(I_n + I_{n-1})$$

D.h. $I_n = -2n\cdot I_n - 2n\cdot I_{n-1} \Rightarrow I_n(2n+1) = -2n\cdot I_{n-1} \Rightarrow I_n = \frac{-2n}{2n+1} I_{n-1}$

$$\Rightarrow I_{n+1} = \frac{-2(n+1)}{2(n+1)+1} I_n$$

lineare, homogene Rekurrenz 1.Ordnung auflösen :

$$I_n = I_0 \cdot \prod_{i=0}^{n-1} \frac{-2(i+1)}{2(i+1)+1} \qquad \left(I_0 = \int_{-1}^{1}(x^2-1)^0 dx = x\Big|_{-1}^{1} = 2 \right)$$

$$I_n = 2\cdot 2^n \cdot n! \cdot \prod_{i=0}^{n-1} \frac{-1}{2(i+1)+1} = 2^{n+1} n!(-1)^n \frac{1}{3\cdot 5\cdot 7\cdot \ldots \cdot (2n+1)}$$

$$= 2^{n+1} n!(-1)^n \frac{2\cdot 4\cdot 6\cdot \ldots \cdot 2n}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot \ldots \cdot (2n+1)} = 2^{n+1} n!(-1)^n \frac{2^n n!}{(2n+1)!} = \frac{(-1)^n 2^{2n+1}(n!)^2}{(2n+1)!}$$

$$\Rightarrow \int_{-1}^{1} f_n(x) f_n(x) dx = (-1)^n (2n)! \int_{-1}^{1}(x^2-1)^n dx = (-1)^n (2n)! \frac{(-1)^n 2^{2n+1}(n!)^2}{(2n+1)!}$$

$$= \frac{2^{2n+1}(n!)^2}{2n+1} = \frac{2^{2n}(n!)^2}{n+\frac{1}{2}}$$

D.h. , für $n = m$ gilt $\int_{-1}^{1}\Phi_n(x)\cdot\Phi_m(x)dx = \int_{-1}^{1}\Phi_n(x)\cdot\Phi_n(x)dx$

$$= \left(n+\frac{1}{2}\right)\cdot\frac{1}{2^n n!}\cdot\frac{1}{2^n n!}\cdot\int_{-1}^{1} f_n(x) f_n(x) dx \quad \text{(siehe (*1))}$$

$$= \frac{n+\frac{1}{2}}{2^{2n}(n!)^2}\cdot\frac{2^{2n}(n!)^2}{n+\frac{1}{2}} = 1$$

Somit wurde bewiesen, daß $\int_{-1}^{1}\Phi_n(x)\cdot\Phi_m(x)dx = d_{n,m}$.

D.h. die $\Phi_n(x)$ bilden im Intervall [-1,+1] ein Orthonormalsystem.

4.2 a)

Fourier-Koeffizienten berechnen :

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos(nx)dx = \lim_{\substack{a\to -p\\ b\to -0}}\left(\frac{1}{p}\int_a^b -\cos(nx)dx\right) + \lim_{\substack{b\to +0\\ c\to +p}}\left(\frac{1}{p}\int_b^c \cos(nx)dx\right)$$

$$= \lim_{\substack{a\to -p\\ b\to -0}}\left(-\frac{1}{p}\frac{\sin(nx)}{n}\Bigg|_a^b\right) + \lim_{\substack{b\to +0\\ c\to +p}}\left(\frac{1}{p}\frac{\sin(nx)}{n}\Bigg|_b^c\right)$$

$$= -\frac{1}{p}\lim_{\substack{a\to -p\\ b\to -0}}\left(\frac{\sin(nb)}{n} - \frac{\sin(na)}{n}\right) + \frac{1}{p}\lim_{\substack{b\to +0\\ c\to +p}}\left(\frac{\sin(nc)}{n} - \frac{\sin(nb)}{n}\right)$$

$$= -\frac{1}{np}\left(\sin(0) - \sin(-np)\right) + \frac{1}{np}\left(\sin(np) - \sin(0)\right)$$

$$= -\frac{1}{np}\sin(np) + \frac{1}{np}\sin(np) \underline{= 0}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin(nx)dx = \lim_{\substack{a\to -p\\ b\to -0}}\left(\frac{1}{p}\int_a^b -\sin(nx)dx\right) + \lim_{\substack{b\to +0\\ c\to +p}}\left(\frac{1}{p}\int_b^c \sin(nx)dx\right)$$

$$= \lim_{\substack{a\to -p\\ b\to -0}}\left(\frac{1}{p}\frac{\cos(nx)}{n}\Bigg|_a^b\right) + \lim_{\substack{b\to +0\\ c\to +p}}\left(-\frac{1}{p}\frac{\cos(nx)}{n}\Bigg|_b^c\right)$$

$$= \frac{1}{p}\lim_{\substack{a\to -p\\ b\to -0}}\left(\frac{\cos(nb)}{n} - \frac{\cos(na)}{n}\right) - \frac{1}{p}\lim_{\substack{b\to +0\\ c\to +p}}\left(\frac{\cos(nc)}{n} - \frac{\cos(nb)}{n}\right)$$

$$= \frac{1}{np}\left(\cos(0) - \cos(-np)\right) - \frac{1}{np}\left(\cos(np) - \cos(0)\right)$$

$$= \frac{1}{np}(1 - \cos(np)) - \frac{1}{np}(\cos(np) - 1) = \frac{1}{np}(2 - 2\cos(np))$$

$$= \frac{2}{np}(1 - \cos(np)) = \frac{2}{np}\left(1 - (-1)^n\right) = \begin{cases} \dfrac{4}{np} & falls\ n\ ungerade \\ 0 & falls\ n\ gerade \end{cases}$$

$$\Rightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$= \sum_{k=1}^{\infty}\frac{4}{(2k-1)p}\sin((2k-1)x) + \sum_{k=1}^{\infty} 0\cdot\sin(2kx) = \frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1}$$

f ist stetig in $(-p,0)\ und\ (0,p)$, dort von beschränkter Variation,

da jeweils $\sum_{j=1}^{n}\left|f(x_j) - f(x_{j-1})\right| = \sum_{j=1}^{n} 0 = 0$ und es gilt

$$f(-p) = \frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\cdot(-p))}{2k-1} = -\frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\cdot p)}{2k-1} = 0$$

$$f(0) = \frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\cdot 0)}{2k-1} = 0$$

$$f(p) = \frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)\cdot p)}{2k-1} = 0$$

D.h. $f(-p) = f(0) = f(p)$

$$\Rightarrow f(x) = \frac{4}{p}\sum_{k=1}^{\infty}\frac{\sin((2k-1)x)}{2k-1} \qquad \forall x \in [-p,p]$$

q.e.d.

4.2 b) erster Teil :
Beweis durch Induktion
???

4.2 b) zweiter Teil :

$$S_n(x) = \frac{4}{p}\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1} \Rightarrow S_n'(x) = \frac{4}{p}\sum_{k=1}^{n}\frac{(2k-1)\cdot\cos(2k-1)x}{2k-1} = \frac{4}{p}\sum_{k=1}^{n}\cos(2k-1)x$$

$$= \frac{4}{p}\frac{\sin(2nt)}{2\sin t} = \frac{2}{p}\frac{\sin(2nt)}{\sin t}$$, nach 4.2b) erster Teil

$$\Rightarrow S_n(x) = \int_0^x S_n'(t)dt = \underline{\frac{2}{p}\int_0^x \frac{\sin(2nt)}{\sin t}dt}$$

4.2 c)

$$S_n'(x) = 0 \Leftrightarrow \frac{\sin(2nx)}{\sin x} = 0 \Leftrightarrow 2nx = kp \qquad k \in N$$

$$\Rightarrow x = \frac{kp}{2n}$$

$$S_n''(x) = \frac{2n\cos(2nx)}{\sin x} - \sin(2nx)\frac{\cos x}{\sin^2 x} = \frac{2n\sin x\cos(2nx) - \sin(2nx)\cos x}{\sin^2 x}$$

$$= \frac{(2n-1)\sin x\cos(2nx) + \sin x\cos(2nx) + \sin(-2nx)\cos x}{\sin^2 x}$$

$$= \frac{(2n-1)\cos(2nx)}{\sin x} - \frac{\sin(2n-1)x}{\sin^2 x}$$

$$\Rightarrow S_n''\left(\frac{kp}{2n}\right) = 2n\cdot\sin\left(\frac{kp}{2n}\right)\cdot\cos(kp) = (-1)^k$$

D.h. die Extrema sind bei $x = \frac{kp}{2n}$ lokales Minimum für gerade k bzw. lokales Maximum für ungerade k

4.2 d)

$$S_n\left(\frac{p}{2n}\right) = \frac{2}{p}\int_0^{\frac{p}{2n}}\frac{\sin(2nt)}{\sin t}dt$$

Setze $2nt = x$ d.h. $t = \frac{x}{2n}$ d.h. $dt = \frac{dx}{2n}$

$$\Rightarrow S_n\left(\frac{p}{2n}\right) = \frac{2}{p}\int_0^p \frac{\sin x}{\sin\left(\frac{x}{2n}\right)}\frac{dx}{2n} = \frac{2}{p}\int_0^p \frac{\sin x}{\frac{\sin\left(\frac{x}{2n}\right)\cdot x}{\frac{x}{2n}}}dx$$

$$\Rightarrow \lim_{n\to\infty} S_n\left(\frac{p}{2n}\right) = \lim_{n\to\infty}\frac{2}{p}\int_0^p \frac{\sin x}{\frac{\sin\left(\frac{x}{2n}\right)\cdot x}{\frac{x}{2n}}}dx$$

zu 4.2 d)

wobei $\lim_{n\to\infty}\left(\frac{\sin\left(\frac{x}{2n}\right)}{\frac{x}{2n}}\right)=1 \qquad \forall x\in(0,p)$

Also $\underline{\lim_{n\to\infty} S_n\left(\frac{p}{2n}\right)=\frac{2}{p}\int_0^p \frac{\sin x}{x}\,dx}$

4.2 e)

Es gilt : $\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-+\ldots$ Taylorsche Entwicklung

$$\Rightarrow \frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-+\ldots$$

Somit ist $S_n\left(\frac{p}{2n}\right)=\frac{2}{p}\int_0^p\frac{\sin x}{x}dx=\frac{2}{p}\left(x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-+\ldots\right)\Big|_0^p$

$$=\frac{2}{p}\left(\left(p-\frac{p^3}{3\cdot 3!}\right)+\left(\frac{p^5}{5\cdot 5!}-\frac{p^7}{7\cdot 7!}\right)+\ldots\right)$$

4.3)

<u>Bemerkung</u> : Es gilt

$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ (*2) sowie

$e^{ix}=\cos x+i\sin x$ (*3)

4.3 a)

Aufgabe : Berechne $\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}$

$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n!}=\sum_{n=0}^{\infty}\frac{\sin(nx)}{n!}$, denn $\frac{\sin(0)}{0!}=0$

mit $z=e^{ix}$ erhält man

$e^z=e^{\left(e^{ix}\right)}=\sum_{n=0}^{\infty}\frac{\left(e^{ix}\right)^n}{n!}$, wegen (*2)

$=\sum_{n=0}^{\infty}\frac{e^{ixn}}{n!}=\sum_{n=0}^{\infty}\frac{\cos(nx)+i\sin(nx)}{n!}$, wegen (*3)

$$=\sum_{n=0}^{\infty}\frac{\cos(nx)}{n!}+i\sum_{n=0}^{\infty}\frac{\sin(nx)}{n!}$$

zu 4.3 a)

$$\Rightarrow \quad \sum_{n=0}^{\infty}\frac{\sin(nx)}{n!} = \operatorname{Im}(e^{z}) = \operatorname{Im}\left(e^{\left(e^{ix}\right)}\right) = \operatorname{Im}\left(e^{\cos x + i\sin x}\right) = \operatorname{Im}\left(e^{\cos x}\cdot e^{i\sin x}\right)$$

$$= \operatorname{Im}\left(e^{\cos x}\left(\cos(\sin x) + i\sin(\sin x)\right)\right) \underline{= e^{\cos x}\cdot\sin(\sin x)}$$

4.3 b)

Aufgabe : Berechne $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)!}$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n-1)x}{(2n-1)!} = \sum_{n=0}^{\infty}(-1)^{n}\frac{\cos(2n+1)x}{(2n+1)!}$$

Es gilt : $\sin z = \sum_{n=0}^{\infty}(-1)^{n}\frac{z^{2n+1}}{(2n+1)!}$ (Analysis I)

mit $z = e^{ix}$ erhält man

$$\sin z = \sin\left(e^{ix}\right) = \sum_{n=0}^{\infty}(-1)^{n}\frac{\left(e^{ix}\right)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n}\frac{e^{ix(2n+1)}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}(-1)^{n}\frac{\cos(2n+1)x + i\sin(2n+1)x}{(2n+1)!} \qquad \text{, wegen (*3)}$$

$$= \sum_{n=0}^{\infty}(-1)^{n}\frac{\cos(2n+1)x}{(2n+1)!} + i\sum_{n=0}^{\infty}(-1)^{n}\frac{\sin(2n+1)x}{(2n+1)!}$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}(-1)^{n}\frac{\cos(2n+1)x}{(2n+1)!} = \operatorname{Re}(\sin z) = \operatorname{Re}\left(\sin e^{ix}\right) = \operatorname{Re}\left(\sin(\cos x + i\sin x)\right)$$

$$= \operatorname{Re}\left(\frac{e^{i(\cos x + i\sin x)} - e^{-i(\cos x + i\sin x)}}{2i}\right) \qquad \text{, wegen } \sin z = \frac{e^{iz}-e^{-iz}}{2i}$$

$$= \operatorname{Re}\left(\frac{1}{2i}\left(e^{i\cos x - \sin x} - e^{\sin x - i\cos x}\right)\right) = \operatorname{Re}\left(\frac{1}{2i}\left(e^{i\cos x}\cdot e^{-\sin x} - e^{\sin x}\cdot e^{-i\cos x}\right)\right)$$

$$= \operatorname{Re}\left(\frac{1}{2i}\left(e^{-\sin x}\left(\cos(\cos x) + i\sin(\cos x)\right) - e^{\sin x}\left(\cos(-\cos x) + i\sin(-\cos x)\right)\right)\right)$$

$$= \operatorname{Re}\left(\frac{1}{2i}\left(e^{-\sin x}\left(\cos(\cos x) + i\sin(\cos x)\right) - e^{\sin x}\left(\cos(\cos x) - i\sin(\cos x)\right)\right)\right)$$

$$= \operatorname{Re}\left(\frac{1}{2i}\left(e^{-\sin x}\cos(\cos x) + e^{-\sin x} i\sin(\cos x) - e^{\sin x}\cos(\cos x) + e^{\sin x} i\sin(\cos x)\right)\right)$$

$$= \operatorname{Re}\left(\begin{array}{l}\frac{1}{2}\left(e^{-\sin x}\sin(\cos x) + e^{\sin x}\sin(\cos x)\right) \\ + \frac{1}{2i}\left(e^{-\sin x}\cos(\cos x) - e^{\sin x}\cos(\cos x)\right)\end{array}\right)$$

$$= \frac{1}{2}\left(e^{-\sin x}\sin(\cos x) + e^{\sin x}\sin(\cos x)\right)$$

$$\underline{= \frac{\sin(\cos x)}{2}\left(e^{\sin x} + e^{-\sin x}\right)}$$